

# Renormalized Energy and Asymptotic Expansion of Optimal Logarithmic Energy on the Sphere

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## Abstract

We study the Hamiltonian of a two-dimensional log-gas with a confining potential  $V$  satisfying the weak growth assumption –  $V$  is of the same order than  $2 \log |x|$  near infinity – considered by Hardy and Kuijlaars [J. Approx. Theory, 170(0) : 44-58, 2013]. We prove an asymptotic expansion, as the number  $n$  of points goes to infinity, for the minimum of this Hamiltonian using the Gamma-Convergence method of Sandier and Serfaty [Ann. Proba., to appear, 2015]. We show that the asymptotic expansion as  $n \rightarrow +\infty$  of the minimal logarithmic energy of  $n$  points on the unit sphere in  $\mathbb{R}^3$  has a term of order  $n$  thus proving a long standing conjecture of Rakhmanov, Saff and Zhou [Math. Res. Letters, 1:647-662, 1994]. Finally we prove the equivalence between the conjecture of Brauchart, Hardin and Saff [Contemp. Math., 578:31-61, 2012] about the value of this term and the conjecture of Sandier and Serfaty [Comm. Math. Phys., 313(3):635-743, 2012] about the minimality of the triangular lattice for a “renormalized energy”  $W$  among configurations of fixed asymptotic density.

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## 1 Introduction

Let  $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$  be a configuration of  $n$  points interacting through a logarithmic potential and confined by an external field  $V$ . The Hamiltonian of this system, also known as a Coulomb gas, is defined as

$$w_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log |x_i - x_j| + n \sum_{i=1}^n V(x_i)$$

where  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^2$ . The minimization of  $w_n$  is linked to the following classical problem of logarithmic potential theory : find a probability measure  $\mu_V$  on  $\mathbb{R}^2$  which minimizes

$$I_V(\mu) := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \frac{V(x)}{2} + \frac{V(y)}{2} - \log |x - y| \right) d\mu(x) d\mu(y) \quad (1.1)$$

amongst all probability measures  $\mu$  on  $\mathbb{R}^2$ . This type of problem dates back to Gauss. More recent references are the thesis of Frostman [11] and the monography of E.Saff and V.Totik [21]. The usual assumptions on  $V : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  are that it is lower semicontinuous, that it is finite on a set of nonzero capacity, and that it satisfies the growth assumption

$$\lim_{|x| \rightarrow +\infty} \{V(x) - 2 \log |x|\} = +\infty. \quad (1.2)$$

These assumptions ensure that a unique minimizer  $\mu_V$  of  $I_V$  exists and that it has compact support.

Recently, Hardy and Kuijlaars [13] (see also [12]) proved that if one replaces (1.2) by the so-called weak growth assumption

$$\liminf_{|x| \rightarrow +\infty} \{V(x) - \log(1 + |x|^2)\} > -\infty, \quad (1.3)$$

then  $I_V$  still admits a unique minimizer, which may no longer have compact support. Moreover Bloom, Levenberg and Wielonsky [1] proved that the classical Frostman type inequalities still hold in this case.

Coming back to the minimum of the discrete energy  $w_n$ , its relation to the minimum of  $I_V$  is that as  $n \rightarrow +\infty$ , the minimum of  $w_n$  is equivalent to  $n^2 \min I_V$ . The next term in the asymptotic expansion of  $w_n$  was derived by Sandier and Serfaty [23] in the classical case (1.2), it reads

$$\min w_n = n^2 \min I_V - \frac{n}{2} \log n + \alpha_V n + o(n),$$

where  $\alpha_V$  is related to the minimum of a Coulombian renormalized energy studied in [22] which quantifies the discrete energy of infinitely many positive charges in the plane screened by a uniform negative background. Note that rather strict assumptions in addition to (1.2) need to be made on  $V$  for this expansion to hold, but they are satisfied in particular if  $V$  is smooth and strictly convex.

Here, we show that such an asymptotic formula still holds when the classical growth assumption (1.2) is replaced with the weak growth assumption (1.3). However it is no longer obvious that the minimum of  $w_n$  is achieved in this case, as the weak growth assumption could allow one point to go to infinity.

**THEOREM 1.1.** *Let  $V$  be an admissible potential<sup>1</sup>. Then the following asymptotic expansion holds.*

$$\inf_{(\mathbb{R}^2)^n} w_n = I_V(\mu_V) n^2 - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx \right) n + o(n), \quad (1.4)$$

where  $\mu_V = m_V(x) dx$  is the unique minimizer of  $I_V$  (see Section 2 for precise definitions of  $W$  and  $\mathcal{A}_1$ .)

This result is proved using the methods in [22, 23] suitably adapted to equilibrium measures with possibly non-compact support together with the compactification approach in [13, 12, 1]. This compactification allows also to connect the discrete energy problem for log gases in the plane with the discrete logarithmic energy problem for finitely many points on the unit sphere  $\mathbb{S}^2$  in the Euclidean space  $\mathbb{R}^3$ .

The logarithmic energy of a configuration  $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$  is given by

$$E_{\log}(y_1, \dots, y_n) := - \sum_{i \neq j}^n \log \|y_i - y_j\|,$$

where  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^3$ . Finding a minimizer of such an energy functional is a problem with many links and ramifications as discussed in the fundamental paper of Saff and Kuijlaars [14] (see also [5]). For instance Smale's 7<sup>th</sup> problem [25] is to find, for any  $n \geq 2$ , a universal constant  $c \in \mathbb{R}$  and a nearly optimal configuration  $(y_1, \dots, y_n) \in (\mathbb{S}^2)^n$  such that, letting  $\mathcal{E}_{\log}(n)$  denote the minimum of  $E_{\log}$  on  $(\mathbb{S}^2)^n$ ,

$$E_{\log}(y_1, \dots, y_n) - \mathcal{E}_{\log}(n) \leq c \log n.$$

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<sup>1</sup>See Section 3.1 for the precise definition

Identifying the term of order  $n$  in the expansion of  $\mathcal{E}_{\log}(n)$  can be seen as a modest step towards a better understanding of this problem.

It was known (lower bound by Wagner [26] and upper bound by Kuijlaars and Saff [15]), that

$$\left(\frac{1}{2} - \log 2\right) n^2 - \frac{1}{2} n \log n + c_1 n \leq \mathcal{E}_{\log}(n) \leq \left(\frac{1}{2} - \log 2\right) n^2 - \frac{1}{2} n \log n + c_2 n$$

for some fixed constant  $c_1$  and  $c_2$ . Thus one can naturally ask for the existence of the limit

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[ \mathcal{E}_{\log}(n) - \left(\frac{1}{2} - \log 2\right) n^2 + \frac{n}{2} \log n \right].$$

**CONJECTURE 1** (Rakhmanov, Saff and Zhou, [20]): There exists a constant  $C$  not depending on  $n$  such that

$$\mathcal{E}_{\log}(n) = \left(\frac{1}{2} - \log 2\right) n^2 - \frac{n}{2} \log n + Cn + o(n) \quad \text{as } n \rightarrow +\infty.$$

**CONJECTURE 2** (Brauchart, Hardin and Saff, [6]): The constant  $C$  in Conjecture 1 is equal to  $C_{BHS}$ , where

$$C_{BHS} := 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}. \quad (1.5)$$

As we will see, our results imply that the last conjecture is equivalent to one concerning the global optimizer of the renormalized energy  $W$ .

**CONJECTURE 3** (Sandier and Serfaty, [22], or see the review by Serfaty [24]) : The triangular lattice is a global minimizer of  $W$  among discrete subsets of  $\mathbb{R}^2$  with asymptotic density one.

The expansion (1.4) in the particular case  $V(x) = \log(1 + |x|^2)$  transported to  $\mathbb{S}^2$  using an inverse stereographic projection and appropriate rescaling gives an expansion for  $\mathcal{E}_{\log}(n)$  and thus proves Conjecture 1. The constant  $C$  in Conjecture 1 can moreover be expressed in terms of the minimum of the renormalized energy  $W$ . The value of  $W$  for the triangular lattice obviously provides an upper bound for this minimum, and by using the Chowla-Selberg formula to compute the expression given in [22] for this quantity, we show that this upper bound is precisely  $C_{BHS}$ . This bound is of course sharp if and only if Conjecture 3 is true. Thus we deduce from (1.4) the following.

**THEOREM 1.2.** *There exists  $C \neq 0$  independent of  $n$  such that, as  $n \rightarrow +\infty$ ,*

$$\mathcal{E}_{\log}(n) = \left(\frac{1}{2} - \log 2\right) n^2 - \frac{n}{2} \log n + Cn + o(n), \quad C = \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2.$$

*Moreover  $C \leq C_{BHS}$  where  $C_{BHS}$  is given in (1.5), and equality holds iff  $\min_{\mathcal{A}_1} W$  is achieved for the triangular lattice of density one.*

The plan of the paper is as follows. In Section 2 we recall the definition of  $W$  and some of its properties from [22]. In Section 3 we recall results about existence, uniqueness and variational Frostman inequalities for  $\mu_V$ . Moreover, we give the precise definition of an admissible potential  $V$ . In Sections 4 and 5 we adapt the method of [22] to the case of equilibrium measures with noncompact support. The expansion (1.4) is proved in Section 6. Finally in Section 7 we prove Conjecture 1 about the existence of  $C$ , the upper bound  $C \leq C_{BHS}$  and the equivalence between Conjectures 2 and 3.

## 2 Renormalized Energy

Here we recall the definition of the renormalized energy  $W$  (see [23] for more details). For any  $R > 0$ ,  $B_R$  denotes the ball centered at the origin with radius  $R$ .

**Definition 2.1.** Let  $m$  be a nonnegative number and  $E$  be a vector-field in  $\mathbb{R}^2$ . We say  $E$  belongs to the **admissible class**  $\mathcal{A}_m$  if

$$\operatorname{div} E = 2\pi(\nu - m) \text{ and } \operatorname{curl} E = 0 \quad (2.1)$$

where  $\nu$  has the form

$$\nu = \sum_{p \in \Lambda} \delta_p, \text{ for some discrete set } \Lambda \subset \mathbb{R}^2, \quad (2.2)$$

and if

$$\frac{\nu(B_R)}{|B_R|} \text{ is bounded by a constant independent of } R > 1.$$

**Remark 2.1.** The real  $m$  is the average density of the points of  $\Lambda$  when  $E \in \mathcal{A}_m$ .

**Definition 2.2.** Let  $m$  be a nonnegative number. For any continuous function  $\chi$  and any vector-field  $E$  in  $\mathbb{R}^2$  satisfying (2.1) where  $\nu$  has the form (2.2) we let

$$W(E, \chi) = \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p \in \Lambda} B(p, \eta)} \chi(x) |E(x)|^2 dx + \pi \log \eta \sum_{p \in \Lambda} \chi(p) \right). \quad (2.3)$$

We use the notation  $\chi_{B_R}$  for positive cutoff functions satisfying, for some constant  $C$  independent of  $R$

$$|\nabla \chi_{B_R}| \leq C, \quad \operatorname{Supp}(\chi_{B_R}) \subset B_R, \quad \chi_{B_R}(x) = 1 \text{ if } d(x, B_R^c) \geq 1. \quad (2.4)$$

where  $d(x, A)$  is the Euclidean distance between  $x$  and set  $A$ .

**Definition 2.3.** The **renormalized energy**  $W$  is defined, for  $E \in \mathcal{A}_m$  and  $\{\chi_{B_R}\}_R$  satisfying (2.4), by

$$W(E) = \limsup_{R \rightarrow +\infty} \frac{W(E, \chi_{B_R})}{|B_R|}.$$

**Remark 2.2.** It is shown in [22, Theorem 1] that the value of  $W$  does not depend on the choice of cutoff functions satisfying (2.4), and that  $W$  is bounded below and admits a minimizer over  $\mathcal{A}_1$ .

Moreover (see [22, Eq. (1.9), (1.12)]), if  $E \in \mathcal{A}_m$ ,  $m > 0$ , then

$$E' = \frac{1}{\sqrt{m}} E(./\sqrt{m}) \in \mathcal{A}_1 \quad \text{and} \quad W(E) = m \left( W(E') - \frac{\pi}{2} \log m \right).$$

In particular

$$\min_{\mathcal{A}_m} W = m \left( \min_{\mathcal{A}_1} W - \frac{\pi}{2} \log m \right), \quad (2.5)$$

and  $E$  is a minimizer of  $W$  over  $\mathcal{A}_m$  if and only if  $E'$  minimizes  $W$  over  $\mathcal{A}_1$ .

In the periodic case, we have the following result [22, Theorem 2], which supports Conjecture 3 above:

**Theorem 2.3.** *The unique minimizer, up to rotation, of  $W$  over Bravais lattices<sup>2</sup> of fixed density  $m$  is the triangular lattice*

$$\Lambda_m = \sqrt{\frac{2}{m\sqrt{3}}} \left( \mathbb{Z}(1, 0) \oplus \mathbb{Z} \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right).$$

This is proved in [22] using the result of Montgomery on minimal theta function [18], we provide below an alternative proof.

*Proof.* For any Bravais lattice  $\Lambda$

$$W(\Lambda) = ah(\Lambda) + b,$$

where  $a > 0$ ,  $b \in \mathbb{R}$  and  $h(\Lambda)$  is the height of the flat torus  $\mathbb{C}/\Lambda$  (see [19, 7, 10] for more details).

Indeed, Osgood, Phillips and Sarnak [19, Section 4, page 205] proved, for  $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ ,  $\tau = a + ib$ , that

$$h(\Lambda) = -\log(b|\eta(\tau)|^4) + C, \quad C \in \mathbb{R},$$

where  $\eta$  is the Dedekind eta function<sup>3</sup>. But from [22] we have

$$W(\Lambda) = -\frac{1}{2} \log \left( \sqrt{2\pi b} |\eta(\tau)|^2 \right) + C,$$

therefore  $W(\Lambda) = ah(\Lambda) + b$ .

Then from [19, Corollary 1(b)], the triangular lattice minimizes  $h$  among Bravais lattices with fixed density, hence the same is true for  $W$ .  $\square$

### 3 Equilibrium Problem in the Whole Plane

In this section we recall results on existence, uniqueness and characterization of the equilibrium measure  $\mu_V$  and we give the definition of the admissible potentials.

#### 3.1 Equilibrium measure, Frostman inequalities and differentiation of $U^{\mu_V}$

**Definition 3.1.** ([1]) *Let  $K \subset \mathbb{R}^2$  be a compact set and let  $\mathcal{M}_1(K)$  be the family of probability measures supported on  $K$ . Then the **logarithmic potential** and the **logarithmic energy** of  $\mu \in \mathcal{M}_1(K)$  are defined as*

$$U^\mu(x) := - \int_K \log |x - y| d\mu(y) \quad \text{and} \quad I_0(\mu) := - \iint_{K \times K} \log |x - y| d\mu(x) d\mu(y).$$

*We say that  $K$  is **log-polar** if  $I_0(\mu) = +\infty$  for any  $\mu \in \mathcal{M}_1(K)$  and we say that a Borel set  $E$  is log-polar if every compact subset of  $E$  is log-polar. Moreover, we say that an assertion holds **quasi-everywhere** (q.e.) on  $A \subset \mathbb{R}^2$  if it holds on  $A \setminus P$  where  $P$  is log-polar.*

**Remark 3.1.** We recall that the Lebesgue measure of a log-polar set is zero.

Now we recall results about the existence, the uniqueness and the characterization of the equilibrium measure  $\mu_V$  proved in [11, 21] for the classical growth assumption (1.2), and by Hardy and Kuijlaars [12, 13] (for existence and uniqueness) and Bloom, Levenberg and Wielonsky [1] (for Frostman type variational inequalities) for weak growth assumption (1.3).

<sup>2</sup>A Bravais lattice of  $\mathbb{R}^2$ , also called “simple lattice” is  $L = \mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$  where  $(\vec{u}, \vec{v})$  is a basis of  $\mathbb{R}^2$ .

<sup>3</sup>See Section 7.3

**Theorem 3.2.** ([11, 21, 12, 13, 1]) *Let  $V$  be a lower semicontinuous function on  $\mathbb{R}^2$  such that  $\{x \in \mathbb{R}^2; V(x) < +\infty\}$  is a non log-polar subset of  $\mathbb{R}^2$  satisfying*

$$\liminf_{|x| \rightarrow +\infty} \{V(x) - \log(1 + |x|^2)\} > -\infty.$$

*Then we have :*

1.  $\inf_{\mu \in \mathcal{M}_1(\mathbb{R}^2)} I_V(\mu)$  is finite, where  $I_V$  is given by (1.1).
2. There exists a unique equilibrium measure  $\mu_V \in \mathcal{M}_1(\mathbb{R}^2)$  with

$$I_V(\mu_V) = \inf_{\mu \in \mathcal{M}_1(\mathbb{R}^2)} I_V(\mu)$$

*and the logarithmic energy  $I_0(\mu_V)$  is finite.*

3. The support  $\Sigma_V$  of  $\mu_V$  is contained in  $\{x \in \mathbb{R}^2; V(x) < +\infty\}$  and  $\Sigma_V$  is not log-polar.
4. Let

$$c_V := I_V(\mu_V) - \int_{\mathbb{R}^2} \frac{V(x)}{2} d\mu_V(x) \quad (3.1)$$

*denote the Robin constant. Then we have the following Frostman variational inequalities :*

$$U^{\mu_V}(x) + \frac{V(x)}{2} \geq c_V \quad \text{q.e. on } \mathbb{R}^2, \quad (3.2)$$

$$U^{\mu_V}(x) + \frac{V(x)}{2} \leq c_V \quad \text{for all } x \in \Sigma_V. \quad (3.3)$$

**Remark 3.3.** In particular we have  $U^{\mu_V}(x) + \frac{V(x)}{2} = c_V$  q.e. on  $\Sigma_V$ .

As explained in [12], the hypothesis of Theorem 3.2 can be usefully transported to the sphere  $\mathcal{S}$  in  $\mathbb{R}^3$  centred at  $(0, 0, 1/2)$  with radius  $1/2$ , by the inverse stereographic projection  $T : \mathbb{R}^2 \rightarrow \mathcal{S}$  defined by

$$T(x_1, x_2) = \left( \frac{x_1}{1 + |x|^2}, \frac{x_2}{1 + |x|^2}, \frac{|x|^2}{1 + |x|^2} \right), \text{ for any } x = (x_1, x_2) \in \mathbb{R}^2.$$

We know that  $T$  is a conformal homeomorphism from  $\mathbb{R}^2$  to  $\mathcal{S} \setminus \{N\}$  where  $N := (0, 0, 1)$  is the North pole of  $\mathcal{S}$ .

The procedure is as follows: Given  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ , we may define (see [12])  $\mathcal{V} : \mathcal{S} \rightarrow \mathbb{R}$  by letting

$$\mathcal{V}(T(x)) = V(x) - \log(1 + |x|^2), \quad \mathcal{V}(N) = \liminf_{|x| \rightarrow +\infty} \{V(x) - \log(1 + |x|^2)\}. \quad (3.4)$$

Then (see [12]),  $V$  satisfies the hypothesis of Theorem 3.2 if and only if  $\mathcal{V}$  is a lowersemicontinuous function on  $\mathcal{S}$  which is finite on a nonpolar set. Therefore, in this case, the minimum of

$$I_{\mathcal{V}}(\mu) := \iint_{\mathcal{S} \times \mathcal{S}} \left( -\log \|x - y\| + \frac{\mathcal{V}(x)}{2} + \frac{\mathcal{V}(y)}{2} \right) d\mu(x) d\mu(y)$$

among probability measures on  $\mathcal{S}$  is achieved. Here  $\|x - y\|$  denotes the euclidean norm in  $\mathbb{R}^3$ . Moreover, still from [12], the minimizer  $\mu_{\mathcal{V}}$  is related to  $\mu_V$  by the following relation

$$\mu_{\mathcal{V}} = T_{\#} \mu_V, \quad (3.5)$$

where  $T_{\#} \mu$  denotes the push-forward of the measure  $\mu$  by the map  $T$ .

**Definition 3.2.** We say that  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is **admissible** if it is of class  $C^3$  and if, defining  $\mathcal{V}$  as above,

1. **(H1)** : The set  $\{x \in \mathbb{R}^2; V(x) < +\infty\}$  is not log-polar and  $\liminf_{|x| \rightarrow +\infty} \{V(x) - \log(1+|x|^2)\} > -\infty$ .
2. **(H2)** : The equilibrium measure  $\mu_{\mathcal{V}}$  is of the form  $m_{\mathcal{V}}(x) \mathbf{1}_{\Sigma_{\mathcal{V}}}(x) dx$ , where  $m_{\mathcal{V}}$  is a  $C^1$  function on  $\mathcal{S}$  and  $dx$  denotes the surface element on  $\mathcal{S}$ , where the function  $m_{\mathcal{V}}$  is bounded above and below by positive constants  $\bar{m}$  and  $\underline{m}$ , and where  $\Sigma_{\mathcal{V}}$  is a compact subset of  $\mathcal{S}$  with  $C^1$  boundary.

**Remark 3.4.** Using (H2) and (3.5), we find that

$$d\mu_V(x) = m_V(x) \mathbf{1}_{\Sigma_V} dx,$$

where  $\Sigma_V = T^{-1}(\Sigma_{\mathcal{V}})$  and

$$m_V(x) = \frac{m_{\mathcal{V}}(T(x))}{(1+|x|^2)^2}. \quad (3.6)$$

Note that  $(1+|x|^2)^{-2}$  is the jacobian of the transformation  $T$ .

## 4 Splitting Formula

Assume  $V$  is admissible. We define as in [23] the blown-up quantities:

$$x' = \sqrt{n}x, \quad m'_V(x') = m_V(x), \quad d\mu'_V(x') = m'_V(x') dx'$$

and we define

$$\zeta(x) := U^{\mu_V}(x) + \frac{V(x)}{2} - c_V, \quad (4.1)$$

where  $c_V$  is the Robin constant given in (3.1). Then by (3.2) and (3.3),  $\zeta(x) = 0$  q.e. in  $\Sigma_V$  and  $\zeta(x) \geq 0$  q.e. in  $\mathbb{R}^2 \setminus \Sigma_V$ .

For  $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ , we define  $\nu_n = \sum_{i=1}^n \delta_{x_i}$  and

$$H_n := -2\pi \Delta^{-1}(\nu_n - n\mu_V) = - \int_{\mathbb{R}^2} \log |\cdot - y| d(\nu_n - n\mu_V)(y) = - \sum_{i=1}^n \log |\cdot - x_i| - nU^{\mu_V}$$

where  $\Delta^{-1}$  is the convolution operator with  $\frac{1}{2\pi} \log |\cdot|$ , hence such that  $\Delta \circ \Delta^{-1} = I_2$  where  $\Delta$  denotes the usual laplacian. Moreover we set

$$H'_n := -2\pi \Delta^{-1}(\nu'_n - \mu'_V) \quad (4.2)$$

where  $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ .

**Lemma 4.1.** Let  $V$  be an admissible potential. Then we have

$$\lim_{R \rightarrow +\infty} \int_{B_R} H_n(x) d\mu_V(x) = \int_{\mathbb{R}^2} H_n(x) d\mu_V(x) \quad \text{and} \quad \lim_{R \rightarrow +\infty} W(\nabla H_n, \mathbf{1}_{B_R}) = W(\nabla H_n, \mathbf{1}_{\mathbb{R}^2}). \quad (4.3)$$

*Proof.* From (3.6) and the bounds on  $m_V$  we get<sup>4</sup>

$$\int_{\mathbb{R}^2} \log(1 + |y|) d\mu_V(y) < +\infty. \quad (4.4)$$

Therefore by the dominated convergence argument given in [17, Theorem 9.1, Chapter 5] (used for the continuity of  $U^{\mu_V}$ ), we have

$$H_n(x) = \sum_{i=1}^n \int_{\mathbb{R}^2} \log\left(\frac{|x-y|}{|x-x_i|}\right) d\mu_V(y). \quad (4.5)$$

It follows that  $H_n(x) = O(|x|^{-1})$  as  $|x| \rightarrow +\infty$  which implies the first equality using (3.6).

The second equality follows from the dominated convergence argument of Mizuta in [16, Theorem 1], because from (4.5) we have  $\nabla H_n(x) = O(|x|^{-2})$  as  $|x| \rightarrow +\infty$  and thus  $|\nabla H_n|^2$  in  $L^1(\mathbb{R}^2)$ .  $\square$

**Lemma 4.2.** *Let  $V$  be admissible. Then, for every configuration  $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ ,  $n \geq 2$ , we have*

$$w_n(x_1, \dots, x_n) = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \sum_{i=1}^n \zeta(x_i). \quad (4.6)$$

*Proof.* We may proceed as in the proof of [23, Lemma 3.1] and make use of the Frostman type inequalities (3.2) and (3.3) and Lemma 4.1. The important point is that, as shown in the proof of the previous lemma, we have  $H_n(x) = O(|x|^{-1})$  and  $\nabla H_n(x) = O(|x|^{-2})$  as  $|x| \rightarrow +\infty$  which implies, exactly like in the compact support case, that

$$\lim_{R \rightarrow +\infty} \int_{\partial B_R} H_n(x) \nabla H_n(x) \cdot \vec{\nu}(x) dx = 0$$

where  $\vec{\nu}(x)$  is the outer unit normal vector at  $x \in \partial B_R$ .  $\square$

## 5 Lower bound

Here we follow the strategy of [23], pointing out the required modifications in the noncompact case.

### 5.1 Mass spreading result and modified density $g$

We have the following result from [23, Proposition 3.4]:

**Lemma 5.1.** *Let  $V$  be admissible and assume  $(\nu, E)$  are such that  $\nu = \sum_{p \in \Lambda} \delta_p$  for some finite subset  $\Lambda \subset \mathbb{R}^2$  and  $\operatorname{div} E = 2\pi(\nu - m_V)$ ,  $\operatorname{curl} E = 0$  in  $\mathbb{R}^2$ . Then, given any  $\rho > 0$  there exists a signed measure  $g$  supported on  $\mathbb{R}^2$  and such that:*

- *there exists a family  $\mathcal{B}_\rho$  of disjoint closed balls covering  $\operatorname{Supp}(\nu)$ , with the sum of the radii of the balls in  $\mathcal{B}_\rho$  intersecting with any ball of radius 1 bounded by  $\rho$ , and such that*

$$g(A) \geq -C(\|m_V\|_\infty + 1) + \frac{1}{4} \int_A |E(x)|^2 \mathbf{1}_{\Omega \setminus \mathcal{B}_\rho}(x) dx, \quad \text{for any } A \subset \mathbb{R}^2,$$

where  $C$  depends only on  $\rho$ ;

---

<sup>4</sup>We note that  $\mu_V$  is an equilibrium measure is sufficient to obtain (4.4), as explained by Mizuta in [17, Theorem 6.1, Chapter 2] or by Bloom, Levenberg and Wielonsky in [1, Lemma 3.2]



- we have

$$dg(x) = \frac{1}{2}|E(x)|^2 dx \quad \text{outside } \bigcup_{p \in \Lambda} B(p, \lambda)$$

where  $\lambda$  depends only on  $\rho$ ;

- for any function  $\chi$  compactly supported in  $\mathbb{R}^2$  we have

$$\left| W(E, \chi) - \int \chi dg \right| \leq CN(\log N + \|m_V\|_\infty) \|\nabla \chi\|_\infty \quad (5.1)$$

where  $N = \#\{p \in \Lambda; B(p, \lambda) \cap \text{Supp}(\nabla \chi) \neq \emptyset\}$  for some  $\lambda$  and  $C$  depending only on  $\rho$ ;

- for any  $U \subset \Omega$

$$\#(\Lambda \cap U) \leq C(1 + \|m_V\|_\infty^2 |\hat{U}| + g(\hat{U})) \quad (5.2)$$

where  $\hat{U} := \{x \in \mathbb{R}^2; d(x, U) < 1\}$ .

**Definition 5.1.** Assume  $\nu_n = \sum_{i=1}^n \delta_{x_i}$ . Letting  $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$  be the measure in blown-up coordinates and  $E_{\nu_n} = \nabla H'_n$ , we denote by  $g_{\nu_n}$  the result of applying the previous proposition to  $(\nu'_n, E_{\nu_n})$ .

The following result [23, Lemma 3.7] connects  $g$  and the renormalized energy.

**Lemma 5.2.** ([23]) For any  $\nu_n = \sum_{i=1}^n \delta_{x_i}$ , we have

$$W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) = \int_{\mathbb{R}^2} dg_{\nu_n}. \quad (5.3)$$

## 5.2 Ergodic Theorem

We adapt the abstract setting in [23, Section 4.1]. We are given a Polish space  $X$ , which is a space of functions, on which  $\mathbb{R}^2$  acts continuously. We denote this action  $(\lambda, u) \rightarrow \theta_\lambda u := u(\cdot + \lambda)$ , for any  $\lambda \in \mathbb{R}^2$  and  $u \in X$ . We assume it is continuous with respect to both  $\lambda$  and  $u$ .

We also define  $T_\lambda^\varepsilon$  and  $T_\lambda$  acting on  $\mathbb{R}^2 \times X$ , by  $T_\lambda^\varepsilon(x, u) := (x + \varepsilon\lambda, \theta_\lambda u)$  and  $T_\lambda(x, u) := (x, \theta_\lambda u)$ .

For a probability measure  $P$  on  $\mathbb{R}^2 \times X$  we say that  $P$  is  $T_{\lambda(x)}$ -invariant if for every function  $\lambda$  of class  $C^1$ , it is invariant under the mapping  $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$ .

We let  $\{f_\varepsilon\}_\varepsilon$ , and  $f$  be measurable functions defined on  $\mathbb{R}^2 \times X$  which satisfy the following properties. For any sequence  $\{x_\varepsilon, u_\varepsilon\}_\varepsilon$  such that  $x_\varepsilon \rightarrow x$  as  $\varepsilon \rightarrow 0$  and such that for any  $R > 0$ ,

$$\limsup_{\varepsilon \rightarrow 0} \int_{B_R} f_\varepsilon(x_\varepsilon + \varepsilon\lambda, \theta_\lambda u_\varepsilon) d\lambda < +\infty,$$

we have

1. (Coercivity)  $\{u_\varepsilon\}_\varepsilon$  has a convergent subsequence;
2. ( $\Gamma$ -liminf) If  $\{u_\varepsilon\}_\varepsilon$  converge to  $u$ , then  $\liminf_{\varepsilon \rightarrow 0} f_\varepsilon(x_\varepsilon, u_\varepsilon) \geq f(x, u)$ .

**Remark 5.3.** In contrast with the compact case we do not have the convergence of  $\{x_\varepsilon\}$ .

Now let  $V$  be an admissible potential on  $\mathbb{R}^2$  and  $\mu_V$  its associated equilibrium measure. We have

**Theorem 5.4.** Let  $V$ ,  $X$ ,  $(f_\varepsilon)_\varepsilon$  and  $f$  be as above. We define

$$F_\varepsilon(u) := \int_{\mathbb{R}^2} f_\varepsilon(x, \theta_{\frac{x}{\varepsilon}} u) d\mu_V(x)$$

Assume  $(u_\varepsilon)_\varepsilon \in X$  is a sequence such that  $F_\varepsilon(u_\varepsilon) \leq C$  for any  $\varepsilon > 0$ . Let  $P_\varepsilon$  be the image of  $\mu_V$  by  $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$ , then:

1.  $(P_\varepsilon)_\varepsilon$  admits a convergent subsequence to a probability measure  $P$ ,
2. the first marginal of  $P$  is  $\mu_V$ ,
3.  $P$  is  $T_{\lambda(x)}$ -invariant,
4. for  $P$ -a.e.  $(x, u)$ ,  $(x, u)$  is of the form  $\lim_{\varepsilon \rightarrow 0} (x_\varepsilon, \theta_{\frac{x_\varepsilon}{\varepsilon}} u_\varepsilon)$ ,
5.  $\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u)$ .
6. Moreover we have

$$\int_{\mathbb{R}^2 \times X} f(x, u) dP(x, u) = \int_{\mathbb{R}^2 \times X} \left( \lim_{R \rightarrow +\infty} \oint_{B_R} f(x, \theta_\lambda u) d\lambda \right) dP(x, u). \quad (5.4)$$

where  $\oint_{B_R}$  denote the integral average over  $B_R$ .

*Proof.* The proof follows [22, 23] but with  $\mu_V$  replacing the normalized Lebesgue measure on a compact set  $\Sigma_V$ . We sketch it and detail the parts where modifications are needed. For any  $R > 0$  we let  $\mu_V^R$  denote the restriction of  $\mu_V$  to  $B_R$ , and  $P_\varepsilon^R$  denote the image of  $\mu_V^R$  by the map  $x \mapsto (x, \theta_{\frac{x}{\varepsilon}} u_\varepsilon)$ .

**STEP 1: Convergence of a subsequence of  $(P_\varepsilon)$  to a probability measure  $P$ .** It suffices to prove that the sequence  $\{P_\varepsilon\}_\varepsilon$  is tight. From [22, 23], which deals with the compact case,  $\{P_\varepsilon^R\}_\varepsilon$  is tight, for any  $R > 0$ .

Now take any  $\delta > 0$ , we need to prove that there exists a compact subset  $K_\delta$  of  $\mathbb{R}^2 \times X$  such that  $P_\varepsilon(K_\delta) > 1 - \delta$  for any  $\varepsilon > 0$ . For this we choose first  $R > 0$  large enough so that  $(\mu_V - \mu_V^R)(\mathbb{R}^2) < \delta/2$ . This implies that  $P_\varepsilon^R$  has total measure at least  $1 - \delta/2$  and then we may use the tightness of  $\{P_\varepsilon^R\}_\varepsilon$  to find that there exists a compact set  $K_\delta$  such that  $P_\varepsilon^R(K_\delta) > 1 - \delta$ . It follows that  $P_\varepsilon(K_\delta) > 1 - \delta$ , and then that  $\{P_\varepsilon\}_\varepsilon$  is tight.

**STEP 2:  $P$  is  $T_{\lambda(x)}$ -invariant.** Let  $\lambda$  be a function of class  $C^1$  on  $\mathbb{R}^2$ ,  $\Phi$  be a bounded continuous function on  $\mathbb{R}^2 \times X$  and  $P_\lambda$  be the image of  $P$  by  $(x, u) \mapsto (x, \theta_{\lambda(x)} u)$ . By the change of variables  $y = \varepsilon \lambda(x) + x = (\varepsilon \lambda + I_2)(x)$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) &= \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(x, \theta_{\lambda(x)} u) dP_\varepsilon(x, u) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\lambda(x) + \frac{x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi(x, \theta_{\frac{\varepsilon \lambda(x) + x}{\varepsilon}} u_\varepsilon) d\mu_V(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{\Phi\left((\varepsilon \lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon\right) m_V((\varepsilon \lambda + I_2)^{-1}(y)) dy}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon \lambda)^{-1}(y)))|}, \end{aligned}$$

where  $D\lambda$  is the differential of  $\lambda$ .

From the boundedness of  $\Phi$  and the decay properties of  $m_V$  (see (3.6)) it is straightforward to show that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{\Phi((\varepsilon\lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon) m_V((\varepsilon\lambda + I_2)^{-1}(y)) dy}{|\det(I_2 + \varepsilon D\lambda((I_2 + \varepsilon\lambda)^{-1}(y)))|} \\ = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2} \Phi((\varepsilon\lambda + I_2)^{-1}(y), \theta_{\frac{y}{\varepsilon}} u_\varepsilon) m_V(y) dy \\ = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi((\varepsilon\lambda + I_2)^{-1}(y), u) dP_\varepsilon(y, u). \end{aligned}$$

Then, arguing as in [23] using the tightness of  $(P_\varepsilon)_\varepsilon$  we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi((\varepsilon\lambda + I_2)^{-1}(y), u) dP_\varepsilon(y, u) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times X} \Phi(y, u) dP_\varepsilon(y, u) \\ &= \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u), \end{aligned}$$

Which concludes the proof that  $\int_{\mathbb{R}^2 \times X} \Phi(x, u) dP_\lambda(x, u) = \int_{\mathbb{R}^2 \times X} \Phi(x, u) dP(x, u)$ , i.e. that  $P$  is  $T_{\lambda(x)}$ -invariant.

Items 2 and 4 in the theorem are obvious consequences of the definition of  $P$  and items 5 and 6. require no modification from [23]. We have proved above items 1 and 3.  $\square$

## 6 Asymptotic Expansion of the Hamiltonian

We define

$$\alpha_V := \frac{1}{\pi} \int_{\mathbb{R}^2} \min_{\mathcal{A}_{m_V(x)}} W dx = \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx, \quad (6.1)$$

where the equality is a consequence of (2.5). The fact that  $\alpha_V$  is finite follows from (3.6), which ensures that the integral converges.

We also let

$$F_n(\nu) = \begin{cases} \frac{1}{n} \left( \frac{1}{\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + 2n \int \zeta d\nu \right) & \text{if } \nu \text{ is of the form } \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \\ +\infty & \text{otherwise.} \end{cases}$$

For  $\nu'_n = \sum_{i=1}^n \delta_{x'_i}$ , let  $E_{\nu_n}$  be a solution of  $\operatorname{div} E_{\nu_n} = 2\pi(\nu'_n - m'_V)$ ,  $\operatorname{curl} E_{\nu_n} = 0$  and we set

$$P_{\nu_n} := \int_{\mathbb{R}^2} \delta_{(x, E_{\nu_n}(x\sqrt{n}+))} d\mu_V(x).$$

The following result extends [23, Theorem 2] to a class of equilibrium measures with possibly unbounded support, which requires a restatement which makes it slightly different from its counterpart in [23]. It is essentially a Gamma-Convergence (see [2]) statement, consisting of a lower bound and an upper bound, the two implying the convergence of  $\frac{1}{n} \left[ w_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right]$  to  $\alpha_V$  for a minimizer  $(x_1, \dots, x_n)$  of  $w_n$ .

## 6.1 Main result

**Theorem 6.1.** *Let  $1 < p < 2$  and  $X = \mathbb{R}^2 \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ . Let  $V$  be an admissible function.*

**A. Lower bound:** *Let  $(\nu_n)_n$  such that  $F_n(\nu_n) \leq C$ , then:*

1.  $P_{\nu_n}$  is a probability measure on  $X$  and admits a subsequence which converges to a probability measure  $P$  on  $X$ ,
2. the first marginal of  $P$  is  $\mu_V$ ,
3.  $P$  is  $T_{\lambda(x)}$ -invariant,
4.  $E \in \mathcal{A}_{m_V(x)}$   $P$ -a.e.,
5. we have the lower bound

$$\liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) \geq \alpha_V. \quad (6.2)$$

**B. Upper bound.** *Conversely, assume  $P$  is a  $T_{\lambda(x)}$ -invariant probability measure on  $X$  whose first marginal is  $\mu_V$  and such that for  $P$ -almost every  $(x, E)$  we have  $E \in \mathcal{A}_{m_V(x)}$ . Then there exist a sequence  $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$  of measures on  $\mathbb{R}^2$  and a sequence  $\{E_n\}_n$  in  $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$  such that  $\operatorname{div} E_n = 2\pi(\nu'_n - m'_V)$  and such that, defining  $P_n = \int_{\mathbb{R}^2} \delta_{(x, E_n(x\sqrt{n}+))} d\mu_V(x)$ , we have  $P_n \rightarrow P$  as  $n \rightarrow +\infty$  and*

$$\limsup_{n \rightarrow +\infty} F_n(\nu_n) \leq \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E). \quad (6.3)$$

**C. Consequences for minimizers.** *If  $(x_1, \dots, x_n)$  minimizes  $w_n$  for every  $n$  and measure  $\nu_n = \sum_{i=1}^n \delta_{x_i}$ , then:*

1. for  $P$ -almost every  $(x, E)$ ,  $E$  minimizes  $W$  over  $\mathcal{A}_{m_V(x)}$ ;
2. we have

$$\lim_{n \rightarrow +\infty} F_n(\nu_n) = \lim_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{W(E)}{m_V(x)} dP(x, E) = \alpha_V, \quad (6.4)$$

hence we obtain the following asymptotic expansion, as  $n \rightarrow +\infty$ :

$$\min_{(\mathbb{R}^2)^n} w_n = I_V(\mu_V)n^2 - \frac{n}{2} \log n + \alpha_V n + o(n). \quad (6.5)$$

## 6.2 Proof of the lower bound

We follow the same lines as in [23, Section 4.2]. Because  $F_n(\nu_n) \leq C$  and (4.6), we have that

$$\frac{1}{n^2} w_n(x_1, \dots, x_n) \rightarrow I_V(\mu_V),$$

therefore  $\nu_n$  converges to  $\mu_V$  (this follows from the results in [12]).

We let  $\nu'_n = \sum_i \delta_{x'_i}$ , and  $E_n, H'_n, g_n$  be as in Definition 5.1.

Let  $\chi$  be a  $C^\infty$  cutoff function with support the unit ball  $B_1$  and integral equal to 1. We define

$$\mathbf{f}_n(x, \nu, E, g) := \begin{cases} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) & \text{if } (\nu, E, g) = \theta_{\sqrt{n}x}(\nu'_n, E_n, g_n), \\ +\infty & \text{otherwise.} \end{cases}$$

As in [23, Section 4.2, Step 1], if we let

$$\mathbf{F}_n(\nu, E, g) := \int_{\mathbb{R}^2} \mathbf{f}_n \left( x, \theta_{x\sqrt{n}}(\nu, E, g) \right) d\mu_V(x), \quad (6.6)$$

then

$$\begin{aligned} \mathbf{F}_n(\nu'_n, E_n, g_n) &= \int_{\mathbb{R}^2} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} d(\theta_{x\sqrt{n}} \# g) d\mu_V(x) = \frac{1}{\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi(y - x\sqrt{n}) dx dg_n(y) \\ &\leq \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + \frac{g_n^-(U^c)}{n\pi}, \end{aligned}$$

by (5.3), where  $U = \{x' : d(x', \mathbb{R}^2 \setminus \Sigma) \geq 1\}$ . As in [23], we have  $g_n^-(U^c) = o(n)$ . Hence, if  $(\nu, E, g) = (\nu'_n, E_n, g_n)$ , as  $n \rightarrow +\infty$ :

$$\mathbf{F}_n(\nu, E, g) \leq \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) + o(1),$$

and  $\mathbf{F}_n(\nu, E, g) = +\infty$  otherwise.

Now, as in [23], we want to use Theorem 5.4 with  $\varepsilon = \frac{1}{\sqrt{n}}$  and  $X = \mathcal{M}_+ \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2) \times \mathcal{M}$  where  $p \in ]1, 2[$ ,  $\mathcal{M}_+$  is the set of nonnegative Radon measures on  $\mathbb{R}^2$  and  $\mathcal{M}$  the set of Radon measures bounded below by  $-C_V := -C(\|m_V\|_\infty^2 + 1)$ . Let  $Q_n$  be the image of  $\mu_V$  by  $x \mapsto (x, \theta_{x\sqrt{n}}(\nu'_n, E_n, g_n))$ . We have:

1) The fact that  $\mathbf{f}_n$  is coercive is proved as in [23, Lemma 4.4]. Indeed, if  $(x_n, \nu(n), E(n), g(n))_n$  is such that  $x_n \rightarrow x$  and, for any  $R > 0$ ,

$$\limsup_{n \rightarrow +\infty} \int_{B_R} \mathbf{f}_n \left( x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu(n), E(n), g(n)) \right) d\lambda < +\infty,$$

then the integrand is bounded for a.e.  $\lambda$ . By assumption on  $\mathbf{f}_n$ , for any  $n$ ,

$$\theta_\lambda(\nu(n), E(n), g(n)) = \theta_{x_n\sqrt{n}+\lambda}(\nu'_n, E_n, g_n),$$

hence it follows that

$$(\nu(n), E(n), g(n)) = \theta_{x_n\sqrt{n}}(\nu'_n, E_n, g_n).$$

For any  $R > 0$ , there exists  $C_R > 0$  such that for any  $n > 0$ , noting  $B_R(x)$  the closed ball of radius  $R$  centred at a point  $x$ ,

$$\begin{aligned} \int_{B_R} \mathbf{f}_n \left( x_n + \frac{\lambda}{\sqrt{n}}, \theta_\lambda(\nu_n, E_n, g_n) \right) d\lambda &= \int_{B_R} \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V \left( x_n + \frac{\lambda}{\sqrt{n}} \right)} d(\theta_{\lambda+x_n\sqrt{n}} \# g_n(y)) d\lambda \\ &= \frac{1}{\pi} \int_{B_R} \int_{\mathbb{R}^2} \frac{\chi(y - x_n\sqrt{n} - \lambda)}{m_V \left( x_n + \frac{\lambda}{\sqrt{n}} \right)} dg_n(y) d\lambda \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \chi * \left( \mathbf{1}_{B_R(x_n\sqrt{n})} \frac{1}{m_V(\cdot/\sqrt{n})} \right) (y) dg_n(y) < C_R. \end{aligned}$$

This, inequalities (3.6) and the fact that  $g_n$  is bounded below imply that  $g_n(B_R(x_n\sqrt{n}))$  is bounded independently of  $n$ . Hence by the same argument as in [23, Lemma 4.4], we have the convergence of a subsequence of  $(\nu_n, E_n, g_n)$ .

2) We have the  $\Gamma$ -liminf property: if  $(x_n, \nu_n, E_n, g_n) \rightarrow (x, \nu, E, g)$  as  $n \rightarrow +\infty$ , then, by Fatou's Lemma,

$$\liminf_{n \rightarrow +\infty} \mathbf{f}_n(x_n, \nu_n, E_n, g_n) \geq f(x, \nu, E, g) := \frac{1}{\pi} \int \frac{\chi(y)}{m_V(x)} dg(y),$$

obviously if the left-hand side is finite. Therefore, Theorem 5.4 applies and implies that:

1. The measure  $Q_n$  admits a subsequence which converges to a measure  $Q$  which has  $\mu_V$  as first marginal.
2. It holds that  $Q$ -almost every  $(x, \nu, E, g)$  is of the form  $\lim_{n \rightarrow +\infty} (x_n, \theta_{x_n \sqrt{n}}(\nu'_n, E_n, g_n))$ .
3. The measure  $Q$  is  $T_{\lambda(x)}$ -invariant.
4. We have  $\liminf_{n \rightarrow +\infty} \mathbf{F}_n(\nu'_n, E_n, g_n) \geq \frac{1}{\pi} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} \frac{\chi(y)}{m_V(x)} dg(y) \right) dQ(x, \nu, E, g)$ .
5.  $\frac{1}{\pi} \int \int \frac{\chi(y)}{m_V(x)} dg(y) dQ(x, \nu, E, g) = \int \left( \lim_{R \rightarrow +\infty} \int_{B_R} \int \frac{\chi(y - \lambda)}{m_V(x)} dg(y) d\lambda \right) dQ(x, \nu, E, g)$ .

Now we can follow exactly the lines of [23, Section 4.2, Step 3] to deduce from 4), after noticing that  $P_n$  is the marginal of  $Q_n$  corresponding to the variables  $(x, E)$  which converge to a  $T_{\lambda(x)}$ -invariant probability measure. Moreover

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n\pi} W(\nabla H'_n, \mathbf{1}_{\mathbb{R}^2}) &\geq \int \left( \int \chi dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &= \int \lim_{R \rightarrow +\infty} \left( \frac{1}{\pi R^2} \int \chi * \mathbf{1}_{B_R} dg \right) \frac{dQ(x, \nu, E, g)}{m_V(x)} \\ &\geq \frac{1}{\pi} \int W(E) \frac{dQ(x, \nu, E, g)}{m_V(x)} = \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP(x, E). \end{aligned}$$

Thus the lower bound (6.2) is proved. The fact that the right-hand side is larger than  $\alpha_V$  is obvious because the first marginal of  $\frac{dP}{m_V}$  is the Lebesgue measure.

### 6.3 Proof of the upper bound, the case $\text{Supp}(\mu_V) \neq \mathbb{R}^2$

The discussion following Theorem 3.2 permits to immediately reduce the case of  $V$ 's such that  $\text{Supp}(\mu_V) \neq \mathbb{R}^2$  to the case of a compact support. Indeed in this case there exists  $y \in \mathcal{S}$  which does not belong to the support of  $\mu_V$ . Let  $R$  be a rotation such that  $R(N) = y$ , then the minimum of  $I_{\mathcal{V} \circ R}$  is  $\mu_{\mathcal{V} \circ R} = R^{-1} \# \mu_V$  hence  $N$  does not belong to its support.

Letting  $\varphi = T^{-1}RT$ , we have that  $\varphi$  is of the form  $z \rightarrow \frac{az+b}{cz+d}$  with  $ad-bc=1$ , and applying (3.4), (3.5) to  $\mathcal{V} \circ R$  we have that

$$\mu_{V_\varphi} = T^{-1} \# \mu_{\mathcal{V} \circ R},$$

where

$$\mathcal{V} \circ R(T(x)) = V_\varphi(x) - \log(1 + |x|^2).$$

This implies that  $\mu_{V_\varphi}$  has compact support since  $N$  does not belong to the support of  $\mu_{\mathcal{V} \circ R}$ . Moreover, using (3.4) again to evaluate  $\mathcal{V}(RT(x))$  we find for any  $x$  such that  $RT(x) \neq N$ , i.e.  $x \neq -d/c$ ,

$$V_\varphi(x) = V(T^{-1}RT(x)) - \log(1 + |T^{-1}RT(x)|^2) + \log(1 + |x|^2),$$

$$V_\varphi(-d/c) = \mathcal{V}(N) + \log(1 + |d/c|^2) = \log(1 + |d/c|^2) + \liminf_{|x| \rightarrow +\infty} \{V(x) - \log(1 + |x|^2)\}.$$

Finally we find that

$$V_\varphi(x) = V(\varphi(x)) - \log(1 + |\varphi(x)|^2) + \log(1 + |x|^2), \quad V_\varphi(-d/c) = \liminf_{y \rightarrow -d/c} V_\varphi(y). \quad (6.7)$$

Now we rewrite the discrete energy by changing variables, to find that, writing  $w_{n,V}$  instead of  $w_n$  to clarify the dependence on  $V$ ,

$$w_{n,V}(x_1, \dots, x_n) = - \sum_{i \neq j}^n \log |\varphi(y_i) - \varphi(y_j)| + n \sum_{i=1}^n V(\varphi(y_i)), \quad (6.8)$$

where  $x_i = \varphi(y_i)$ . Now we use the identity (see [12])

$$\|T(x) - T(y)\| = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}$$

applied to  $\varphi(x)$ ,  $\varphi(y)$  together with the fact that  $\varphi = T^{-1}RT$  and that  $R$  is a rotation to get

$$\|T(x) - T(y)\| = \frac{|\varphi(x) - \varphi(y)|}{\sqrt{1 + |\varphi(x)|^2} \sqrt{1 + |\varphi(y)|^2}}.$$

The two together imply that

$$\log |\varphi(x) - \varphi(y)| = \log |x - y| + \frac{1}{2} \log(1 + |\varphi(x)|^2) + \frac{1}{2} \log(1 + |\varphi(y)|^2) - \frac{1}{2} \log(1 + |x|^2) - \frac{1}{2} \log(1 + |y|^2).$$

Replacing in (6.8) shows that

$$w_{n,V}(x_1, \dots, x_n) = w_{n,V_\varphi}(y_1, \dots, y_n) + \sum_i \log(1 + |\varphi(y_i)|^2) - \sum_i \log(1 + |y_i|^2), \quad x_i = \varphi(y_i). \quad (6.9)$$

It follows from (6.9) that an upper bound for  $\min w_{n,V}$  can be computed by using a minimizer for  $w_{n,V_\varphi}$  as a test function. But now we recall that  $\mu_{V_\varphi}$  has compact support, hence the results of [23] apply and we find, using the fact that for such a minimizer  $\frac{1}{n} \sum_i \delta_{y_i}$  converges to  $\mu_{V_\varphi}$ ,

$$\min w_{n,V} \leq n^2 I_{V_\varphi}(\mu_{V_\varphi}) - \frac{1}{2} n \log n + \left( \alpha_{V_\varphi} + \int \log \left( \frac{1 + |\varphi(x)|^2}{1 + |x|^2} \right) d\mu_{V_\varphi}(x) \right) n + o(n), \quad (6.10)$$

where

$$\alpha_{V_\varphi} = \frac{\alpha_1}{\pi} - \frac{1}{2} \int_{\Sigma_{V_\varphi}} m_{V_\varphi}(x) \log m_{V_\varphi}(x) dx, \quad \alpha_1 := \min_{\mathcal{A}_1} W.$$

We remark that  $I_{V_\varphi}(\mu_{V_\varphi}) = I_V(\mu_V)$  because  $\mu_{V_\varphi} = \varphi^{-1} \# \mu_V$ . Moreover, it follows from (3.6) that

$$m_{V_\varphi}(x) = m_V(\varphi(x)) \left( \frac{1 + |\varphi(x)|^2}{1 + |x|^2} \right)^2,$$

which plugged in the expression for  $\alpha_{V_\varphi}$  and then in (6.10) yields,

$$\min w_{n,V} \leq n^2 I_V(\mu_V) - \frac{1}{2} n \log n + \alpha_V n + o(n),$$

which matches the lower-bound we already obtained and thus proves Theorem 1.1 in the case where the support of  $\mu_V$  is not the full plane.

## 6.4 Proof of the upper bound by compactification and conclusion

Here we assume that  $\Sigma_V = \mathbb{R}^2$ . Let

$$\varphi(z) := -\frac{1}{z} = \varphi^{-1}(z).$$

Then, using the notations of the previous section, we deduce from (6.7) that

$$V_\varphi(z) = V(\varphi(z)) + 2 \log |z|.$$

To simplify exposition and notation, we assume that  $\mu_V(B_1) = \mu_V(B_1^c) = 1/2$ , otherwise there would exist  $R$  such that  $\mu_V(B_R) = \mu_V(B_R^c) = 1/2$  and we should use the transformation  $\varphi_R(z) = \varphi_R^{-1}(z) = -Rz^{-1}$  instead.

Our idea is to cut  $\Sigma_V = \mathbb{R}^2$  into two parts in order to construct a sequence of  $2n$  points associated to a sequence of vector-fields. We will only construct test configurations with an even number of points, again to simplify exposition and avoid unessential technicalities.

**STEP 1: Reminder of the compact case and notations.** We need [23, Corollary 4.6] when  $K$  is a compact set of  $\mathbb{R}^2$ :

**Theorem 6.2.** ([23]) *Let  $P$  be a  $T_{\lambda(x)}$ -invariant probability measure on  $X = K \times L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$ , where  $K$  is a compact subset of  $\mathbb{R}^2$  with  $C^1$  boundary.*

*We assume that  $P$  has first marginal  $dx|_K/|K|$  and that for  $P$ -almost every  $(x, E)$  we have  $E \in \mathcal{A}_{m(x)}$ , where  $m$  is a smooth function on  $K$  bounded above and below by positive constants. Then there exists a sequence  $\{\nu_n = \sum_{i=1}^n \delta_{x_i}\}_n$  of empirical measures on  $K$  and a sequence  $\{E_n\}_n$  in  $L_{loc}^p(\mathbb{R}^2, \mathbb{R}^2)$  such that  $\text{div } E_n = 2\pi(\nu'_n - m')$ , such that  $E_n = 0$  outside  $K$  and such that  $P_n := \int_K \delta_{(x, E_n(\sqrt{n}x+))} dx \rightarrow P$  as  $n \rightarrow +\infty$ . Moreover*

$$\limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(E_n, \mathbf{1}_{\mathbb{R}^2}) \leq \frac{|K|}{\pi} \int W(E) dP(x, E).$$

We write  $\mu_V = \mu_V^1 + \mu_V^2$  where  $\mu_V^1 := \mu_V|_{B_1}$  and  $\mu_V^2 := \mu_V|_{\bar{B}_1^c}$ , where  $\bar{A}$  denotes the closure of set  $A$  in  $\mathbb{R}^2$ . Let  $\tilde{\mu}_V^2 := \varphi\#\mu_V^2$ , then we have

$$d\mu_V^1(x) = m_V(x)\mathbf{1}_{B_1}(x)dx =: m_V^1(x)dx \quad \text{and} \quad d\tilde{\mu}_V^2(x) = m_{V_\varphi}(x)\mathbf{1}_{B_1}(x)dx =: m_{V_\varphi}^2(x)dx,$$

where  $m_{V_\varphi}(x) = m_V(\varphi^{-1}(x))|\det(D\varphi_x^{-1})|$ .

Note that, by assumption **(H2)** and (3.6) we have that there exists positive constants  $\bar{m}$  and  $\underline{m}$  such that, for any  $x \in B_1$ ,

$$0 < \underline{m} \leq m_V(x) \leq \bar{m} \quad \text{and} \quad 0 < \underline{m} \leq m_{V_\varphi}(x) \leq \bar{m}.$$

Moreover the boundary  $\partial B_1$  is  $C^1$ .

Now let  $P$  be a  $T_{\lambda(x)}$ -invariant probability measure on  $X$  whose first marginal is  $\mu_V$  and be such that for  $P$ -almost every  $(x, E)$ , we have  $E \in \mathcal{A}_{m_V(x)}$ . We can write

$$P = P^1 + P^2,$$

where  $P^1$  is the restriction of  $P$  to  $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$  with first marginal  $\mu_V^1$ , and  $P^2$  is the restriction of  $P$  to  $B_1^c \times L_{loc}^p(B_1^c, \mathbb{R}^2)$  with first marginal  $\mu_V^2$ . We define  $\tilde{P}^1$  by the relation

$$dP^1(x, u) = m_V(x)|B_1|d\tilde{P}^1(x, u),$$



and then  $\tilde{P}^1$  is a  $T_{\lambda(x)}$ -invariant probability measure on  $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$  with first marginal  $dx|_{B_1}/|B_1|$  and such that, for  $\tilde{P}^1$ -a.e.  $(x, E)$ ,  $E \in \mathcal{A}_{m_V^1(x)}$ . We denote by  $\varphi\#P^2$  the pushforward of  $P^2$  by

$$(x, E) \mapsto (y, \tilde{E}), \quad \text{where } y := \varphi(x) \text{ and } \tilde{E} := (D\varphi_y)^T E (D\varphi_y \cdot), \quad (6.11)$$

where  $D\varphi_x = \lambda(x)I_2$  is the differential of  $\varphi$  at point  $x$ . Then if  $\operatorname{div} E = 2\pi(\nu - m_V(x)dx)$  then  $\operatorname{div} \tilde{E} = 2\pi(\varphi\#\nu - \lambda^2(y)m_V(\varphi(y)))$  so that for  $\varphi\#P^2$ -a.e.  $(y, \tilde{E})$  the vector field  $\tilde{E}$  belongs to  $\mathcal{A}_{m_{V\varphi}(y)}$ , since

$$m_{V\varphi}(y) dy = m_V(\varphi(y)) d(\varphi(y)) = m_V(\varphi(y)) \lambda(y)^2 dy.$$

We define  $\tilde{P}^2$  by the relation

$$d(\varphi\#P^2)(y, \tilde{E}) = m_{V\varphi}(y)|B_1|d\tilde{P}^2(y, \tilde{E}),$$

and then  $\tilde{P}^2$  is a  $T_{\lambda(x)}$ -invariant probability measure on  $B_1 \times L_{loc}^p(B_1, \mathbb{R}^2)$  with first marginal  $dy|_{B_1}/|B_1|$  and such that, for  $\tilde{P}^2$  a.e.  $(y, \tilde{E})$ ,  $\tilde{E} \in \mathcal{A}_{m_{V\varphi}(y)}$ .

**STEP 2: Application of Theorem 6.2.** We may now apply Theorem 6.2 to  $\tilde{P}^1$  and  $\tilde{P}^2$ . We thus construct a sequence  $\{\nu_n^1 := \sum_{i=1}^n \delta_{x_i^1}\}$  of empirical measures on  $B_1$  and a sequence  $\{E_n^1\}_n$  in  $L_{loc}^p(B_1, \mathbb{R}^2)$  such that

$$\operatorname{div} E_n^1 = 2\pi((\nu_n^1)' - (m_V^1)') \quad \text{and} \quad \tilde{P}_n^1 := \int_{B_1} \delta_{(x, E_n^1(\sqrt{n}x+))} dx \rightarrow \tilde{P}^1,$$

as  $n \rightarrow +\infty$ . Moreover, we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(E_n^1, \mathbf{1}_{\mathbb{R}^2}) \leq \frac{|B_1|}{\pi} \int W(E) d\tilde{P}^1(x, E). \quad (6.12)$$

Applying now the same Theorem to  $\tilde{P}^2$ , we construct a sequence  $\{\tilde{\nu}_n^2 := \sum_{i=1}^n \delta_{\tilde{x}_i^2}\}$  of empirical measures on  $B_1$  and a sequence  $\{\tilde{E}_n^2\}_n$  in  $L_{loc}^p(B_1, \mathbb{R}^2)$  such that

$$\operatorname{div} \tilde{E}_n^2 = 2\pi((\tilde{\nu}_n^2)' - (m_{V\varphi}^2)') \quad \text{and} \quad \tilde{P}_n^2 := \int_{B_1} \delta_{(x, \tilde{E}_n^2(\sqrt{n}x+))} dx \rightarrow \tilde{P}^2,$$

as  $n \rightarrow +\infty$ . Moreover, we have

$$\limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(\tilde{E}_n^2, \mathbf{1}_{\mathbb{R}^2}) \leq \frac{|B_1|}{\pi} \int W(\tilde{E}) d\tilde{P}^2(y, \tilde{E}). \quad (6.13)$$

**STEP 3: Construction of sequences and conclusion.** It is not difficult to see that we can assume  $\tilde{x}_j^2 \neq 0$  for any  $j$  and any  $n \geq 2$  (otherwise we translate a little bit the point). Now we set  $x_j^2 := \varphi(\tilde{x}_j^2)$  and in view of (6.11), for each  $n$  we define

$$\nu_n^2 := \varphi\#\tilde{\nu}_n^2 = \sum_{j=1}^n \delta_{x_j^2} \quad \text{and} \quad E_n^2(x) := (D\varphi_{n^{-1/2}x})^T \tilde{E}_n^2(n^{1/2}\varphi(n^{-1/2}x)).$$

Hence, we have a sequence of vector-fields  $E_n^2$  of  $L_{loc}^p(B_1^c, \mathbb{R}^2)$  such that

$$\operatorname{div} E_n^2 = 2\pi((\nu_n^2)' - (m_V^2)')$$

where  $m_V^2(x) = m_V(x)\mathbf{1}_{\bar{B}_1^c}(x)$  is the density of  $\mu_V^2$ .

We have, for sufficiently small  $\eta$  such that  $0 \notin B(\tilde{x}_i^2, \eta)$  for every  $i$ ,

$$\begin{aligned}
& W(E_n^2, \mathbf{1}_{\mathbb{R}^2}) \\
&= \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(x_i^2, \eta)} |E_n^2(x')|^2 dx' + \pi n \log \eta \right) \\
&= \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(x_i^2, \eta)} |(D\varphi_{n^{-1/2}x'})^T \tilde{E}_n^2(n^{1/2}\varphi(n^{-1/2}x'))|^2 dx' + \pi n \log \eta \right) \\
&= \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(y_i^2, |\varphi'(x_i^2)|\eta)} |\tilde{E}_n^2(y')|^2 dy' + \pi n \log \eta \right) \\
&= \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_{i=1}^n B(y_i^2, |\varphi'(x_i^2)|\eta)} |\tilde{E}_n^2(y')|^2 dy' + \pi \sum_{i=1}^n \log |\varphi'(x_i)|\eta - \pi \sum_{i=1}^n \log |\varphi'(x_i)| \right) \\
&= W(\tilde{E}_n^2, \mathbf{1}_{\mathbb{R}^2}) - \pi \sum_{i=1}^n \log |\varphi'(x_i)|,
\end{aligned}$$

where the change of variable is  $y' = n^{1/2}\varphi(n^{-1/2}x')$ .

Furthermore, we have

$$\int W(\tilde{E}) d\tilde{P}^2(y, \tilde{E}) = \frac{1}{|B_1|} \int W(\tilde{E}) \frac{d(\varphi^\# P^2)(y, \tilde{E})}{m_{V_\varphi}(y)} = \frac{1}{|B_1|} \int W(D\varphi_y^T E(D\varphi_y \cdot)) \frac{dP^2(x, E)}{m_{V_\varphi}(y)}$$

by change of variable  $y = \varphi(x)$  and  $\tilde{E} = D\varphi_y^T E(D\varphi_y \cdot)$ .

Now we remark that, for  $\lambda > 0$  and  $E \in \mathcal{A}_m$ ,

$$\begin{aligned}
W(\lambda E(\lambda \cdot)) &= \lim_{R \rightarrow +\infty} \frac{1}{\pi R^2} \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_i B(y_i, \eta)} \chi_R(y) \lambda^2 |E(\lambda y)|^2 dy + \pi \sum_i \chi_R(y_i) \log \eta \right) \\
&= \lim_{R \rightarrow +\infty} \frac{1}{\pi R^2} \lim_{\eta \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_i B(x_i, \lambda \eta)} \chi_R(x/\lambda) |E(x)|^2 dx + \pi \sum_i \chi_R(x_i/\lambda) \log \eta \right)
\end{aligned}$$

where  $x = \lambda y$ . Thus, setting  $R' = R\lambda$  and  $\eta' = \eta\lambda$ , we get

$$\begin{aligned}
& W(\lambda E(\lambda \cdot)) \\
&= \lim_{R' \rightarrow +\infty} \frac{\lambda^2}{\pi R'^2} \lim_{\eta' \rightarrow 0} \left( \frac{1}{2} \int_{\mathbb{R}^2 \setminus \bigcup_i B(x_i, \eta')} \chi_{R'}(x) |E(x)|^2 dx + \pi \sum_i \chi_{R'}(x_i) (\log \eta' - \log \eta) \right) \\
&= \lambda^2 (W(E) - m \log \lambda).
\end{aligned}$$

Applying this equality with  $\lambda = |\varphi'(x)|^{-1} = |\varphi'^{-1}(y)|$ , we obtain

$$\begin{aligned}
& \limsup_{n \rightarrow +\infty} \frac{1}{n\pi} \left( W(E_n^2, \mathbf{1}_{\mathbb{R}^2}) + \pi \sum_{i=1}^n \log |\varphi'(x_i)| \right) \\
&\leq \frac{1}{\pi} \int \frac{1}{|\varphi'(x)|^2} (W(E) + \log |\varphi'(x)| m_V^2(x)) \frac{dP^2(x, E)}{m_{V_\varphi}(y)},
\end{aligned}$$

that is to say, because  $m_V^2$  is the density of points  $\{x_i\}$  as  $n \rightarrow +\infty$ ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(E_n^2, \mathbf{1}_{\mathbb{R}^2}) + \int_{B_1^c} \log |\varphi'(x)| d\mu_V^2(x)$$

$$\leq \frac{1}{\pi} \int W(E) \frac{dP^2(x, E)}{m_V(x)} + \int_{B_1^c} \log |\varphi'(x)| dP^2(x).$$

As  $\int_{B_1^c} \log |\varphi'(x)| dP^2(x) = \int_{B_1^c} \log |\varphi'(x)| d\mu_V^2(x)$ , it follows that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(E_n^2, \mathbf{1}_{\mathbb{R}^2}) \leq \frac{1}{\pi} \int W(E) \frac{dP^2(x, E)}{m_V(x)}. \quad (6.14)$$

Finally, we set

$$\nu_{2n} := \nu_n^1 + \nu_n^2 \quad \text{and} \quad E_{2n} := E_n^1 + E_n^2,$$

and by (6.12) and (6.14), we have, since  $E_n^1$  and  $E_n^2$  have disjoint supports,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \frac{1}{n\pi} W(E_n, \mathbf{1}_{\mathbb{R}^2}) &\leq \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP^1(x, E) + \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP^2(x, E) \\ &= \frac{1}{\pi} \int \frac{W(E)}{m_V(x)} dP(x, E) \end{aligned}$$

which proves (6.3). Furthermore, by changes of variable,

$$P_n^1 := \int_{B_1} \delta_{(x, E_n^1(x\sqrt{n}+))} d\mu_V(x) \rightarrow P^1 \quad \text{and} \quad P_n^2 := \int_{B_1^c} \delta_{(x, E_n^2(x\sqrt{n}+))} d\mu_V(x) \rightarrow P^2$$

in the weak sense of measure, and it follows that

$$P_n = P_n^1 + P_n^2 \rightarrow P^1 + P^2 = P.$$

Part **C** follows from **A** and **B** as in [23].

## 7 Consequence: the Logarithmic Energy on the Sphere

As we have an asymptotic expansion of the minimum of the Hamiltonian  $w_n$  where minimizers can be in the whole plane – not only in a compact set as in the classical case – we will use the inverse stereographic projection from  $\mathbb{R}^2$  to a sphere in order to determine the asymptotic expansion of optimal logarithmic energy on sphere.

### 7.1 Inverse stereographic projection

Here we recall properties of the inverse stereographic projection used by Hardy and Kuijlaars [12, 13] and by Bloom, Levenberg and Wielonsky [1] in order to prove Theorem 3.2.

Let  $\mathcal{S}$  be the sphere of  $\mathbb{R}^3$  centred at  $(0, 0, 1/2)$  of radius  $1/2$ ,  $\Sigma$  be an unbounded closed set of  $\mathbb{R}^2$  and  $T : \mathbb{R}^2 \rightarrow \mathcal{S}$  be the associated inverse stereographic projection defined by

$$T(x_1, x_2) = \left( \frac{x_1}{1 + |x|^2}, \frac{x_2}{1 + |x|^2}, \frac{|x|^2}{1 + |x|^2} \right), \quad \text{for any } x = (x_1, x_2) \in \mathbb{R}^2,$$

where  $\mathbb{R}^2 := \{(x_1, x_2, 0); x_1, x_2 \in \mathbb{R}\}$ . We know that  $T$  is a conformal homeomorphism from  $\mathbb{C}$  to  $\mathcal{S} \setminus \{N\}$  where  $N := (0, 0, 1)$  is the North pole of  $\mathcal{S}$ .

We have the following identity:

$$\|T(x) - T(y)\| = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad \text{for any } x, y \in \mathbb{R}^2.$$

Furthermore, if  $|y| \rightarrow +\infty$ , we obtain, for any  $x \in \mathbb{R}^2$ :

$$\|T(x) - N\| = \frac{1}{\sqrt{1 + |x|^2}}. \quad (7.1)$$

We note  $\Sigma_{\mathcal{S}} = T(\Sigma) \cup \{N\}$  the closure of  $T(\Sigma)$  in  $\mathcal{S}$ . Let  $\mathcal{M}_1(\Sigma)$  be the set of probability measures on  $\Sigma$ . For  $\mu \in \mathcal{M}_1(\Sigma)$ , we denote by  $T\sharp\mu$  its push-forward measure by  $T$  characterized by

$$\int_{\Sigma_{\mathcal{S}}} f(z) dT\sharp\mu(z) = \int_{\Sigma} f(T(x)) d\mu(x),$$

for every Borel function  $f : \Sigma_{\mathcal{S}} \rightarrow \mathbb{R}$ . The following result is proved in [12]:

**Lemma 7.1.** *The correspondance  $\mu \rightarrow T\sharp\mu$  is a homeomorphism from the space  $\mathcal{M}_1(\Sigma)$  to the set of  $\mu \in \mathcal{M}_1(\Sigma_{\mathcal{S}})$  such that  $\mu(\{N\}) = 0$ .*

## 7.2 Asymptotic expansion of the optimal logarithmic energy on the unit sphere

An important case is the equilibrium measure associated to the potential

$$V(x) = \log(1 + |x|^2)$$

corresponding to the external field  $\mathcal{V} \equiv 0$  on  $\mathcal{S}$  and where  $T\sharp\mu_V$  is the uniform probability measure on  $\mathcal{S}$  (see [12]). Hence  $V$  is an admissible potential and from (3.6) we find

$$d\mu_V(x) = \frac{dx}{\pi(1 + |x|^2)^2} \quad \text{and} \quad \Sigma_V = \mathbb{R}^2.$$

We define

$$\bar{w}_n(x_1, \dots, x_n) := - \sum_{i \neq j}^n \log |x_i - x_j| + (n-1) \sum_{i=1}^n \log(1 + |x_i|^2),$$

and we recall that the logarithmic energy of a configuration  $(y_1, \dots, y_n) \in \mathcal{S}^n$  is given by

$$E_{\log}(y_1, \dots, y_n) := - \sum_{i \neq j}^n \log \|y_i - y_j\|.$$

Furthermore, we recall that  $\mathcal{E}_{\log}(n)$  denotes the minimal logarithmic energy of  $n$  points on  $\mathbb{S}^2$ .

**Lemma 7.2.** *For any  $(x_1, \dots, x_n) \in (\mathbb{R}^2)^n$ , we have the following equalities:*

$$\bar{w}_n(x_1, \dots, x_n) = E_{\log}(T(x_1), \dots, T(x_n)) \quad \text{and} \quad w_n(x_1, \dots, x_n) = E_{\log}(T(x_1), \dots, T(x_n), N),$$

which imply that

$$\begin{aligned} (x_1, \dots, x_n) \text{ minimizes } \bar{w}_n &\iff (T(x_1), \dots, T(x_n)) \text{ minimizes } E_{\log} \\ (x_1, \dots, x_n) \text{ minimizes } w_n &\iff (T(x_1), \dots, T(x_n), N) \text{ minimizes } E_{\log}. \end{aligned}$$

*Proof.* For any  $1 \leq i \leq n$ , we set  $y_i := T(x_i)$ , hence we get, by (7.1),

$$\begin{aligned} E_{\log}(y_1, \dots, y_n) &:= - \sum_{i \neq j}^n \log \|y_i - y_j\| \\ &= - \sum_{i \neq j}^n \log \|T(x_i) - T(x_j)\| \end{aligned}$$

$$\begin{aligned}
&= - \sum_{i \neq j}^n \log \left( \frac{|x_i - x_j|}{\sqrt{1 + |x_i|^2} \sqrt{1 + |x_j|^2}} \right) \\
&= - \sum_{i \neq j}^n \log |x_i - x_j| + (n-1) \sum_{i=1}^n \log(1 + |x_i|^2) \\
&= \bar{w}_n(x_1, \dots, x_n).
\end{aligned}$$

Furthermore, by (7.1), we obtain

$$\begin{aligned}
w_n(x_1, \dots, x_n) &= \bar{w}_n(x_1, \dots, x_n) + \sum_{i=1}^n \log(1 + |x_i|^2) \\
&= - \sum_{i \neq j} \log \|y_i - y_j\| - 2 \sum_{i=1}^n \log \|y_i - N\| = E_{\log}(y_1, \dots, y_n, N).
\end{aligned}$$

□

**Lemma 7.3.** *If  $(x_1, \dots, x_n)$  minimizes  $w_n$  or  $\bar{w}_n$ , then, for  $\nu_n := \sum_{i=1}^n \delta_{x_i}$ , we have*

$$\frac{\nu_n}{n} \rightarrow \mu_V, \quad \text{as } n \rightarrow +\infty,$$

*in the weak sense of measures.*

*Proof.* Let  $(x_1, \dots, x_n)$  be a minimizer of  $\bar{w}_n$ , then  $(T(x_1), \dots, T(x_n))$  is a minimizer of  $E_{\log}$ . Brauchart, Dragnev and Saff proved in [4, Proposition 11] that

$$\frac{1}{n} \sum_{i=1}^n \delta_{T(x_i)} \rightarrow T\#\mu_V.$$

As  $T\#\mu_V(N) = 0$ , by Lemma 7.1 we get the result.

If  $(x_1, \dots, x_n)$  is a minimizer of  $w_n$ , then  $(T(x_1), \dots, T(x_n), N)$  minimizes  $E_{\log}$  and we can use our previous argument because

$$\frac{1}{n+1} \left( \sum_{i=1}^n \delta_{T(x_i)} + \delta_N \right) = \frac{1}{n} \sum_{i=1}^n \delta_{T(x_i)} \left( \frac{n}{n+1} \right) + \frac{\delta_N}{n+1} \rightarrow T\#\mu_V,$$

in the weak sense of measures, and we have the same conclusion. □

**Lemma 7.4.** *If  $(x_1, \dots, x_n)$  is a minimizer of  $w_n$  and if  $\nu_n := \sum_{i=1}^n \delta_{x_i}$  then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + |x|^2) d\nu_n(x) = \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x).$$

*There exists minimizers of  $\bar{w}_n$  for which the same is true.*

*Proof.* Let  $(x_1, \dots, x_n)$  be a minimizer of  $\bar{w}_n$ . We define  $y_i := T(x_i)$  for any  $1 \leq i \leq n$  and we notice that

$$\frac{1}{n} \int_{\mathbb{R}^2} \log(1 + |x|^2) d\nu_n(x) = -\frac{2}{n} \int_{\mathbb{R}^2} \log \left( \frac{1}{\sqrt{1 + |x|^2}} \right) d\nu_n(x)$$

$$= -\frac{2}{n} \int_{\mathcal{S}} \log \|y - N\| dT \# \nu_n(y),$$

and by the previous Lemma,  $(y_1, \dots, y_n)$  is a minimizer of  $E_{\log}$  on  $\mathcal{S}$ . For any rotation  $R$  of  $\mathcal{S}$  the rotated configuration of points is still a minimizer, and it is clear that the average over rotations  $R$  of

$$-\frac{2}{n} \sum_i \log \|Ry_i - N\|$$

is equal to

$$-2 \int_{\mathcal{S}} \log \|y - N\| dy.$$

It follows that there exists a rotated configuration  $(\bar{y}_1, \dots, \bar{y}_n)$  such that

$$\frac{1}{n} \sum_i \log \|\bar{y}_i - N\| = \int_{\mathcal{S}} \log \|y - N\| dy.$$

Transporting this equality back to  $\mathbb{R}^2$  with  $T^{-1}$ , we obtain a minimizer  $(\bar{x}_1, \dots, \bar{x}_n)$  of  $\bar{w}_n$  such that

$$\frac{1}{n} \sum_i \log(1 + |\bar{x}_i|^2) = \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x).$$

If  $(x_1, \dots, x_n)$  is a minimizer of  $w_n$  we use [4, Theorem 15] about the optimal point separation which yields the existence of constants  $C$  and  $n_0$  such that for any  $n \geq n_0$  and any minimizer  $\{y_1, \dots, y_n\} \in \mathcal{S}^n$  of the logarithmic energy on the sphere, we have

$$\min_{i \neq j} \|y_i - y_j\| > \frac{C}{\sqrt{n-1}}.$$

Letting  $y_i = T(x_i)$  we have that  $(N, y_1, \dots, y_n)$  is a minimizer of the logarithmic energy, hence for any  $1 \leq i \leq n$ ,

$$\|y_i - N\| > \frac{C}{\sqrt{n-1}}. \quad (7.2)$$

For  $n \geq n_0$  and  $\delta > 0$  sufficiently small, we define, for any  $0 < r \leq \delta$ ,

$$n(r) := \# \{y_i \mid y_i \in B(N, r) \cap \mathcal{S}\},$$

and  $r_i = \|y_i - N\|$ . From the separation property there exists a constant  $C$  such that  $n(r) \leq Cr^2n$  for any  $r$ . Hence we have, using integration by parts,

$$\begin{aligned} - \sum_{y_i \in B(N, \delta)} \log r_i &= - \int_{1/\sqrt{n-1}}^{\delta} \log rn'(r) dr \\ &= -n(\delta) \log \delta + \int_{1/\sqrt{n-1}}^{\delta} \frac{n(r)}{r} dr \\ &\leq -Cn\delta^2 \log \delta + Cn \int_{1/\sqrt{n-1}}^{\delta} r dr \leq C\delta^2 n |\log \delta|. \end{aligned}$$

It follows that

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B(N, \delta) \cap \mathcal{S}} \log \|y - N\| dT \# \nu_n(y) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow +\infty} - \sum_{i=1}^{n_\delta} \frac{1}{n} \log \|y_i - N\| = 0. \quad (7.3)$$

By Lemma 7.3,  $\frac{\nu_n}{n}$  goes weakly to the measure  $\mu_V$  on  $B_R$  for any  $R$ , hence we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + |x|^2) d\nu_n(x) \\ &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left( \int_{B_R} \log(1 + |x|^2) d\nu_n(x) + \int_{B_R^c} \log(1 + |x|^2) d\nu_n(x) \right) \\ &= \int_{B_R} \log(1 + |x|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + |x|^2) d\nu_n(x). \end{aligned}$$

Therefore it follows from (7.3) that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\mathbb{R}^2} \log(1 + |x|^2) d\nu_n(x) \\ &= \lim_{R \rightarrow +\infty} \left( \int_{B_R} \log(1 + |x|^2) d\mu_V(x) + \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{B_R^c} \log(1 + |x|^2) d\nu_n(x) \right) \\ &= \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x). \end{aligned}$$

The convergence is proved.  $\square$

The following result proves the existence of the constant  $C$  in the Conjecture 1 of Rakhmanov, Saff and Zhou.

**Theorem 7.5.** *We have*

$$\mathcal{E}_{\log}(n) = \left( \frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 \right) n + o(n), \quad \text{as } n \rightarrow +\infty.$$

*Proof.* As  $E_{\log}$  is invariant by translation of the 2-sphere, we work on the sphere  $\tilde{\mathbb{S}}^2$  of radius 1 and centred in  $(0, 0, 1/2)$ . Let  $(y_1, \dots, y_n) \in \tilde{\mathbb{S}}^2$  be a minimizer of  $E_{\log}$ . Without loss of generality, for any  $n$ , we can choose this configuration such that  $y_i \neq N$  for any  $1 \leq i \leq n$ . Hence there exists  $(x_1, \dots, x_n)$  such that  $\frac{y_i}{2} = T(x_i)$  for any  $i$  and we get

$$\begin{aligned} E_{\log}(y_1, \dots, y_n) &= - \sum_{\substack{i,j \\ i \neq j}}^n \log \|y_i - y_j\| \\ &= - \sum_{\substack{i,j \\ i \neq j}}^n \log \|T(x_i) - T(x_j)\| - n(n-1) \log 2 \\ &= \bar{w}_n(x_1, \dots, x_n) - n(n-1) \log 2. \end{aligned}$$

By Lemma 7.2,  $(y_1, \dots, y_n)$  is a minimizer of  $E_{\log}$  if and only if  $(x_1, \dots, x_n)$  is a minimizer of  $\bar{w}_n$ . By the lower bound (6.2) and the convergence of Lemma 7.4, we have, for some minimizer  $(\bar{x}_1, \dots, \bar{x}_n)$  of  $\bar{w}_n$ :

$$\begin{aligned} & \liminf_{n \rightarrow +\infty} \frac{1}{n} \left[ \bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &= \liminf_{n \rightarrow +\infty} \frac{1}{n} \left[ w_n(\bar{x}_1, \dots, \bar{x}_n) - \sum_{i=1}^n \log(1 + |\bar{x}_i|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\ &\geq \alpha - \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x). \end{aligned}$$

The upper bound (6.3) and Lemma 7.3 yield,  $(x_1, \dots, x_n)$  being a minimizer of  $w_n$ :

$$\begin{aligned}
& \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[ \bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
& \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[ \bar{w}_n(x_1, \dots, x_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
& = \limsup_{n \rightarrow +\infty} \frac{1}{n} \left[ w_n(x_1, \dots, x_n) - \sum_{i=1}^n \log(1 + |x_i|^2) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] \\
& = \alpha_V - \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x).
\end{aligned}$$

Thus, we get

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[ \bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) - n^2 I_V(\mu_V) + \frac{n}{2} \log n \right] = \alpha_V - \int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x).$$

Therefore, we have the following asymptotic expansion, as  $n \rightarrow +\infty$ , when  $(\bar{x}_1, \dots, \bar{x}_n)$  is a minimizer of  $\bar{w}_n$ :

$$\begin{aligned}
& \bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) \\
& = n^2 I_V(\mu_V) - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W - \frac{1}{2} \int_{\mathbb{R}^2} m_V(x) \log m_V(x) dx - \int_{\mathbb{R}^2} V(x) d\mu_V(x) \right) n + o(n).
\end{aligned}$$

We know that  $I_V(\mu_V) = \frac{1}{2}$  (see [3, Eq. (2.26)]) and

$$\begin{aligned}
\int_{\mathbb{R}^2} \log(1 + |x|^2) d\mu_V(x) & = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\log(1 + |x|^2)}{(1 + |x|^2)^2} \\
& = 2 \int_0^{+\infty} \frac{r \log(1 + r^2)}{(1 + r^2)^2} dr \\
& = - \left[ \frac{\log(1 + r^2)}{1 + r^2} \right]_0^{+\infty} + \int_{\mathbb{R}^2} \frac{2r}{(1 + r^2)^2} dr \\
& = - \left[ \frac{1}{1 + r^2} \right]_0^{+\infty} \\
& = 1.
\end{aligned}$$

Hence we obtain, as  $n \rightarrow +\infty$ ,

$$\begin{aligned}
\bar{w}_n(\bar{x}_1, \dots, \bar{x}_n) & = \frac{n^2}{2} - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{1}{2} \int \log(\pi(1 + |x|^2)^2) d\mu_V(x) - 1 \right) n + o(n) \\
& = \frac{n^2}{2} - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \int \log(1 + |x|^2) d\mu_V(x) - 1 \right) n + o(n) \\
& = \frac{n^2}{2} - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} \right) n + o(n),
\end{aligned}$$

and the asymptotic expansion of  $\mathcal{E}_{\log}(n)$  is

$$\mathcal{E}_{\log}(n) = \left( \frac{1}{2} - \log 2 \right) n^2 - \frac{n}{2} \log n + \left( \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 \right) n + o(n).$$

□



**Remark 7.6.** It follows from lower bound proved by Rakhmanov, Saff and Zhou [20, Theorem 3.1], that

$$\begin{aligned} \frac{1}{\pi} \min_{\mathcal{A}_1} W + \frac{\log \pi}{2} + \log 2 &= \lim_{n \rightarrow +\infty} \frac{1}{n} \left[ E_{\log}(y_1, \dots, y_n) - \left( \frac{1}{2} - \log 2 \right) n^2 + \frac{n}{2} \log n \right] \\ &\geq -\frac{1}{2} \log \left[ \frac{\pi}{2} (1 - e^{-a})^b \right], \end{aligned}$$

where  $a := \frac{2\sqrt{2\pi}}{\sqrt{27}} \left( \sqrt{2\pi + \sqrt{27}} + \sqrt{2\pi} \right)$  and  $b := \frac{\sqrt{2\pi + \sqrt{27}} - \sqrt{2\pi}}{\sqrt{2\pi + \sqrt{27}} + \sqrt{2\pi}}$ , and we get

$$\min_{\mathcal{A}_1} W \geq -\frac{\pi}{2} \log \left[ 2\pi^2 (1 - e^{-a})^b \right] \approx -4.6842707.$$

### 7.3 Computation of renormalized energy for the triangular lattice and upper bound for the term of order $n$

Sandier and Serfaty proved in [22, Lemma 3.3] that

$$W(\Lambda_{1/2\pi}) = -\frac{1}{2} \log \left( \sqrt{2\pi b} |\eta(\tau)|^2 \right),$$

where  $\Lambda_{1/2\pi}$  is the triangular lattice corresponding to the density  $m = 1/2\pi$ ,  $\tau = a + ib = 1/2 + i\frac{\sqrt{3}}{2}$  and  $\eta$  is the Dedekind eta function defined, with  $q = e^{2i\pi\tau}$ , by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

We recall Chowla-Selberg formula (see [8] or [9, Proposition 10.5.11] for details):

$$\prod_{m=1}^{|D|} \Gamma \left( \frac{m}{|D|} \right)^{\frac{w}{2} \left( \frac{D}{m} \right)} = 4\pi \sqrt{-D} b |\eta(\tau)|^4,$$

for  $\tau$  a root of the integral quadratic equation  $\alpha z^2 + \beta z + \gamma = 0$  where  $D = \beta^2 - 4\alpha\gamma < 0$ ,  $\left( \frac{D}{m} \right)$  is the Kronecker symbol,  $w$  the number of roots of unity in  $\mathbb{Q}(i\sqrt{-D})$  and when the class number of  $\mathbb{Q}(i\sqrt{-D})$  is equal to 1. In our case  $b = \sqrt{3}/2$ ,  $w = 6$ ,  $\alpha = \beta = \gamma = 1$  because  $\tau$  is a root of unity, hence  $D = -3$ ,  $\left( \frac{-3}{1} \right) = 1$  and  $\left( \frac{-3}{2} \right) = -1$  by the Gauss Lemma. Therefore we obtain

$$\Gamma(1/3)^3 \Gamma(2/3)^{-3} = 4\pi \sqrt{3} \frac{\sqrt{3}}{2} |\eta(\tau)|^4,$$

and by Euler's reflection formula  $\Gamma(1 - 1/3) \Gamma(1/3) = \frac{\pi}{\sin(\pi/3)}$ , we get

$$\frac{\Gamma(1/3)^6 3\sqrt{3}}{8\pi^3} = \frac{4\pi \sqrt{3} \times \sqrt{3} |\eta(\tau)|^4}{2}.$$

Finally we obtain

$$|\eta(\tau)|^4 = \frac{\Gamma(1/3)^6 \sqrt{3}}{16\pi^4}.$$

Now it is possible to find the exact value of the renormalized energy of the triangular lattice  $\Lambda_1$  of density  $m = 1$ :

$$\begin{aligned}
W(\Lambda_1) &= 2\pi W(\Lambda_{1/2\pi}) - \pi \frac{\log(2\pi)}{2} \\
&= -\pi \log \left( \sqrt{2\pi b} |\eta(\tau)|^2 \right) - \pi \frac{\log(2\pi)}{2} \\
&= \pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2}\pi \log 2 \\
&= \pi \log \left( \frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3} \right) \\
&\approx -4.1504128.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\frac{1}{\pi} W(\Lambda_1) + \frac{\log \pi}{2} + \log 2 \\
&= \frac{1}{\pi} \left( \pi \log \pi - \frac{\pi}{2} \log 3 - 3\pi \log(\Gamma(1/3)) + \frac{3}{2}\pi \log 2 \right) + \frac{\log \pi}{2} + \log 2 \\
&= 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)} = C_{BHS} \approx -0.0556053,
\end{aligned}$$

and we find exactly the value  $C_{BHS}$  conjectured by Brauchart, Hardin and Saff in [6, Conjecture 4]. Therefore Conjecture 2 is true if and only if the triangular lattice  $\Lambda_1$  is a global minimizer of  $W$  among vector-fields in  $\mathcal{A}_1$ , i.e.

$$\min_{\mathcal{A}_1} W = W(\Lambda_1) = \pi \log \left( \frac{2\sqrt{2}\pi}{\sqrt{3}\Gamma(1/3)^3} \right).$$

Thus we obtain the following result

**Theorem 7.7.** *We have:*

1. *It holds*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \left[ \mathcal{E}_{\log}(n) - \left( \frac{1}{2} - \log 2 \right) n^2 + \frac{n}{2} \log n \right] \leq 2 \log 2 + \frac{1}{2} \log \frac{2}{3} + 3 \log \frac{\sqrt{\pi}}{\Gamma(1/3)}.$$

2. *Conjectures 2 and 3 are equivalent, i.e.  $\min_{\mathcal{A}_1} W = W(\Lambda_1) \iff C = C_{BHS}$ .*

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